SECOND EIGENVALUE OF PANEITZ OPERATORS AND MEAN CURVATURE

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Abstract. For $n \geq 7$, we give the optimal estimate for the second eigenvalue of Paneitz operators for compact n-dimensional submanifolds in an $(n+p)$-dimensional space form.

1. introduction

Assume that M^n is a compact Riemannian manifold immersed into Euclidean space $\mathbb{R}^{n+p}$. In [9], Reilly obtained the estimates for the first eigenvalue λ_1 of Laplacian

$$\lambda_1 \leq \frac{n}{V(M^n)} \int_{M^n} |H|^2, \tag{1.1}$$

where H is the mean curvature vector of immersion M^n in $\mathbb{R}^{n+p}$, $V(M^n)$ is the volume of M^n. In [11], El Soufi and Ilias obtained the corresponding estimates for submanifolds in unit sphere $\mathbb{S}^{n+p}(1)$, hyperbolic space $\mathbb{H}^{n+p}(-1)$ and some other ambient spaces. Motivated by [7], El Soufi and Ilias [12] obtained the sharp estimates for the second eigenvalue of Schrödinger operator for compact submanifolds M^n in space form $\mathbb{R}^{n+p}$, $\mathbb{S}^{n+p}(1)$ and hyperbolic space $\mathbb{H}^{n+p}(-1)$.

Given a smooth 4-dimensional Riemannian manifold (M^4, g), the Paneitz operator, discovered in [8], is the fourth-order operator defined by

$$P^4 f = \Delta^2 f - \mathrm{div}\Big(\frac{2}{3}R\ \mathrm{Id} - 2Ric\Big)df, \qquad \text{for} \quad f \in C^\infty(M^4),$$

where Δ is the scalar Laplacian defined by $\Delta = \mathrm{div}d$, div is the divergence with respect to g, R, Ric are the scalar curvature and Ricci curvature respectively. The Paneitz operator was generalized to higher dimensions by Branson [1]. Given a smooth compact Riemannian n-manifold (M^n, g), $n \geq 5$, let P be the operator defined by (see also [2])

$$Pf = \Delta^2 f - \mathrm{div}\Big(a_n R\ \mathrm{Id} + b_n Ric\Big)df + \frac{n-4}{2}Qf, \tag{1.2}$$

where

$$\begin{aligned} Q &= c_n|Ric|^2 + d_n R^2 - \frac{1}{2(n-1)}\Delta R \\ &= \frac{n^2-4}{8n(n-1)^2}R^2 - \frac{2}{(n-2)^2}|E|^2 - \frac{1}{2(n-1)}\Delta R, \qquad E = Ric - \frac{R}{n}g, \end{aligned} \tag{1.3}$$

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and

$$a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}, \qquad b_n = -\frac{4}{n-2}, \qquad c_n = -\frac{2}{(n-2)^2} \qquad d_n = \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}. \tag{1.4}$$

The operator P is also called Paneitz operator (or Branson-Paneitz operator).

In [2, 4, 5, 14], the authors investigated positivity of the Paneitz operator. In analogy with the conformal volume in [7], Xu and Yang [14] defined N-conformal energy for compact 4-dimensional Riemannian manifold immersed in N-dimensional sphere $\mathbb{S}^N(1)$. In the same paper [14], the upper bound for the first eigenvalue of Paneitz operator was bounded by using n-conformal energy. In [3], we obtained the sharp estimates for the first eigenvalue of Paneitz operator for compact 4-dimensional submanifolds in Euclidean space and unit sphere.

The aim of this paper is to obtain the optimal estimates for the second eigenvalue of Paneitz operator in terms of the extrinsic geometry of the compact submanifold M^n in space form $R^{n+p}(c)$(the Euclidean space $\mathbb{R}^{n+p}$ for $c=0$, the Euclidean unit sphere $\mathbb{S}^{n+p}(1)$ for $c=1$ and the hyperbolic space $\mathbb{H}^{n+p}(-1)$ for $c=-1$). Considering the first eigenvalue Λ_1 of P, it is easy to see that it is bounded by the mean value of the Q-curvature on M,

$$\Lambda_1 \leq \frac{n-4}{2V(M^n)} \int_{M^n} Q dv_g. \tag{1.5}$$

Moreover, the inequality (1.5) is strict unless Q is constant.

For the second eigenvalue we have the following

Theorem 1.1. *Let $\phi : M^n \to R^{n+p}(c)$ be an n-dimensional ($n \geq 7$) compact submanifold. Then the second eigenvalue λ_2 of Paneitz operator satisfies*

$$\Lambda_2 V(M^n) \leq \frac{1}{2} n(n^2-4) \int_{M^n} \left(|H|^2 + c\right)^2 dv_g + \frac{n-4}{2} \int_{M^n} Q dv_g. \tag{1.6}$$

Moreover, the equality holds if and only if $\phi(M^n)$ is an n-dimensional geodesic sphere $\mathbb{S}^n(r_c)$ in $R^{n+p}(c)$, where

$$r_0 = \frac{1}{2}\left(\frac{n(n+4)(n^2-4)}{\Lambda_2}\right)^{1/4}, \quad r_1 = arcsin\ r_0, \quad r_{-1} = sinh^{-1} r_0. \tag{1.7}$$

Remark 1.1. For the n-dimensional geodesic sphere $\mathbb{S}^n(r_c)$ in $R^{n+p}(c)$, we have

$$\begin{aligned} \Lambda_1 &= \frac{1}{16} n(n-4)(n^2-4)(|H|^2+c)^2 \\ \Lambda_2 &= \frac{1}{16} n(n+4)(n^2-4)(|H|^2+c)^2 \\ Q &= \frac{1}{8} n(n^2-4)\left(|H|^2+c\right)^2. \end{aligned}$$

From (1.3) and (1.6), we can reach

Corollary 1.2. *Under the same assumptions as in the theorem 1.1, then*

$$\Lambda_2 V(M^n) \leq \frac{1}{2}n(n^2-4)\int_{M^n}\Big(|H|^2+c\Big)^2 dv_g + \frac{(n-4)(n^2-4)}{16n(n-1)^2}\int_{M^n} R^2. \tag{1.8}$$

Moreover, the equality holds if and only if M^n is an n-dimensional geodesic sphere.

Remark 1.2. We note that our technique in proof of Theorem 1.1 does not work for $3 \leq n \leq 6$, so it is interesting to know that Theorem 1.1 is true or not for $3 \leq n \leq 6$.

2. Some lemmas

Assume that $\phi : M^n \to R^{n+p}(c)$ is an n-dimensional compact submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$. From [7](see also [12]), it is known that

Lemma 2.1. *Let w be the first eigenfunction of Paneitz operator P on M^n. Then there exists a regular conformal map*

$$\Gamma : R^{n+p}(c) \to \mathbb{S}^{n+p}(1) \subset R^{n+p+1} \tag{2.1}$$

such that for all $1 \leq \alpha \leq n+p+1$, the immersion $X = \Gamma \circ \phi = (X^1, \cdots, X^{n+p+1})$ satisfies

$$\int_{M^n} X^\alpha w dv_g = 0$$

where g is induced metric of $\phi : M^n \longrightarrow R^{n+p}(c)$.

Assume that $\tilde{g} = e^{2u}g$ is a conformal transformation for $u \in C^\infty(M)$, then the scalar curvature obeys [10]

$$e^{2u}\tilde{R} = R - 2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2, \tag{2.2}$$

the gradient operator and the Laplacian follows

$$\tilde{\Delta} f = e^{-2u}\left[\Delta f + (n-2)\nabla u \cdot \nabla f\right], \tag{2.3}$$

$$\tilde{\nabla} f = e^{-u}\nabla f, \qquad e^{2u}|\tilde{\nabla} f|^2 = |\nabla f|^2, \tag{2.4}$$

where ∇ and Δ (resp. $\tilde{\nabla}$ and $\tilde{\Delta}$) are the Levi-Civita connection and Laplacian with respect to g (resp. $\tilde{g}$).

We have the following relation under conformal transformation $\tilde{g} = e^{2u}g$ (see page 766 in [12]),

Lemma 2.2. *Let $\phi : M^n \to R^{n+p}(c)$ be an n-dimensional submanifold and $X = \Gamma \circ \phi$ as before. Then we have*

$$e^{2u}(|\tilde{\boldsymbol{h}}|^2 - n|\tilde{H}|^2) = |\boldsymbol{h}|^2 - n|H|^2, \tag{2.5}$$

where $\boldsymbol{h}, \tilde{\boldsymbol{h}}$ are the second fundamental form of the immersion ϕ and X respectively, $H = \frac{1}{n}tr\boldsymbol{h}$ and $\tilde{H} = \frac{1}{n}tr\tilde{\boldsymbol{h}}$ are the mean curvature vectors, u is defined by

$$e^{2u} = \frac{1}{n}|\nabla(\Gamma \circ \phi)|^2. \tag{2.6}$$

We need also the following result (see also [12]):

Lemma 2.3. *Let $\phi : M \to R^{n+p}(c)$ be an n-dimensional submanifold and $X = \Gamma \circ \phi$ as before. Then we have*

$$e^{2u}\left(|\tilde{H}|^2 + 1\right) = |H|^2 + c - \frac{2}{n}\Delta u - \frac{n-2}{n}|\nabla u|^2. \tag{2.7}$$

Proof. The Gauss equation for $\phi : M \to R^{n+p}(c)$ states

$$|\mathbf{h}|^2 - n|H|^2 = n(n-1)|H|^2 + n(n-1)c - R, \tag{2.8}$$

Similarly,

$$|\tilde{\mathbf{h}}|^2 - n|\tilde{H}|^2 = n(n-1)|\tilde{H}|^2 + n(n-1) - \tilde{R}, \tag{2.9}$$

Combining (2.5), (2.8) and (2.9), we get

$$n(n-1)(|H|^2 + c) - R = \left[n(n-1)\left(|\tilde{H}|^2 + 1\right) - \tilde{R}\right] e^{2u}, \tag{2.10}$$

i.e.

$$n(n-1)e^{2u}(|\tilde{H}|^2 + 1) = n(n-1)(|H|^2 + c) - [R - e^{2u}\tilde{R}] \tag{2.11}$$

From (2.2), we have

$$R - e^{2u}\tilde{R} = 2(n-1)\Delta u + (n-1)(n-2)|\nabla u|^2,$$

i.e.

$$\frac{1}{n(n-1)}(R - e^{2u}\tilde{R}) = \frac{2}{n}\Delta u + \frac{n-2}{n}|\nabla u|^2. \tag{2.12}$$

Inserting (2.12) into (2.11) yields (2.7). □

The following lemma is crucial in the proof of our Theorem 1.1.

Lemma 2.4. *Let $\phi : M \to R^{n+p}(c)$ be an n-dimensional compact submanifold, $X = \Gamma \circ \phi$ as before and u be defined by (2.6), then*

$$\int_{M^n} e^{2u}(|H|^2 + c) \le \int_{M^n} (|H|^2 + c)^2 - \frac{n-6}{n}\int_{M^n} e^{2u}|\nabla u|^2. \tag{2.13}$$

Proof. Multiplying e^{2u} in both sides of (2.7), we have

$$e^{4u}(|\tilde{H}|^2 + 1) = e^{2u}(|H|^2 + c) - \frac{2}{n}e^{2u}\Delta u - \frac{n-2}{n}e^{2u}|\nabla u|^2. \tag{2.14}$$

Integrating (2.14) over M and noting

$$\int_{M^n} e^{2u}\Delta u = -2\int_{M^n} e^{2u}|\nabla u|^2,$$

we can get

$$\int_{M^n} e^{4u} \le \int_{M^n} e^{2u}(|H|^2 + c) - \frac{n-6}{n}\int_{M^n} e^{2u}|\nabla u|^2. \tag{2.15}$$

From the Cauchy-Schwartz inequality and (2.15), we have

$$\begin{aligned} 2\int_{M^n} e^{2u}(|H|^2 + c) &\le \int_{M^n} e^{4u} + \int_{M^n} (|H|^2 + c)^2 \\ &\le \int_{M^n} e^{2u}(|H|^2 + c) - \frac{n-6}{n}\int_{M^n} e^{2u}|\nabla u|^2 + \int_{M^n} (|H|^2 + c)^2. \end{aligned}$$

This inequality implies (2.13). $\square$

3. Proof of Theorem 1.1

Assume that $\phi : M^n \to R^{n+p}(c)$ be an n-dimensional compact submanifold in an $(n+p)$-dimensional space form $R^{n+p}(c)$. From Lemma 2.1, there exists a regular conformal map

$$\Gamma : R^{n+p}(c) \to \mathbb{S}^{n+p}(1) \subset R^{n+p+1}$$

such that the immersion $X = \Gamma \circ \phi = (X^1, \cdots, X^{n+p+1})$ satisfies

$$\int_{M^n} X^\alpha w dv_g = 0, \qquad \text{for all } 1 \leq \alpha \leq n+p+1,$$

where w is the first eigenfunction of the Paneitz operator on M^n.

Let Λ_2 be the second eigenvalue of Paneitz operator P. From the max-min principle for the Paneitz operator, we have

$$\Lambda_2 \int_{M^n} (X^\alpha)^2 dv_g \leq \int_{M^n} P(X^\alpha) \cdot X^\alpha dv_g, \qquad 1 \leq \alpha \leq n+p+1. \tag{3.1}$$

Making summation over α from 1 to $n+p$ in (3.1), using the fact $\sum_{\alpha=1}^{n+p+1} (X^\alpha)^2 = 1$ and (1.2), we can obtain

$$\begin{aligned}
\Lambda_2 V(M) &\leq \sum_{\alpha=1}^{n+p+1} \int_{M^n} P(X^\alpha) \cdot X^\alpha dv_g \\
&= \int_{M^n} \Big[\sum_{\alpha=1}^{n+p+1} \Delta^2 X^\alpha \cdot X^\alpha - \sum_{j,k=1}^{n} < \{(a_n R \delta_{jk} + b_n R_{jk}) X_j\}_k, X > \\
&\quad + \frac{n-4}{2} \int_{M^n} Q|X|^2 \Big] dv_g \\
&= \int_{M^n} < \Delta X, \Delta X > dv_g + \int_{M^n} \sum_{j,k=1}^{n} < (a_n R \delta_{jk} + b_n R_{jk}) X_j, X_k > \\
&\quad + \frac{n-4}{2} \int_{M^n} Q|X|^2 dv_g
\end{aligned} \tag{3.2}$$

where we use Stokes' formula in the second equality.

By (2.3) and (2.4), we have the following calculations

$$\begin{aligned}
&< \Delta X, \Delta X > \\
&= e^{4u} < \tilde{\Delta} X - (n-2) \tilde{\nabla} u \cdot \tilde{\nabla} X, \tilde{\Delta} X - (n-2) \tilde{\nabla} u \cdot \tilde{\nabla} X > \\
&= e^{4u} < n\tilde{H} - nX - (n-2) \tilde{\nabla} u \cdot \tilde{\nabla} X, n\tilde{H} - nX - (n-2) \tilde{\nabla} u \cdot \tilde{\nabla} X > \\
&= e^{4u} [n^2 |\tilde{H}|^2 + n^2 + (n-2)^2 |\tilde{\nabla} u|^2] \\
&= e^{2u} [n^2 e^{2u} |\tilde{H}|^2 + n^2 e^{2u} + (n-2)^2 |\nabla u|^2],
\end{aligned} \tag{3.3}$$

where $\tilde{H}$ is the mean curvature vector of $X = \Gamma \circ \phi : M^n \longrightarrow \mathbb{S}^{n+p}(1)$, here we used in the second equality the following well-known formula $\tilde{\Delta} X = n\tilde{H} - nX$.

Noting

$$< X_j, X_k >= e^{2u}\delta_{jk}, \tag{3.4}$$

and putting (3.3) into (3.2), we have

$$\begin{aligned}\Lambda_2 V(M) \leq & \int_{M^n} e^{2u}\Big[n^2 e^{2u}\left(|\tilde{H}|^2 + 1\right) + (n-2)^2|\nabla u|^2\Big]dv_g \\ & + (na_n + b_n)\int_{M^n} Re^{2u}dv_g + \frac{n-4}{2}\int_{M^n} Qdv_g.\end{aligned} \tag{3.5}$$

Putting (2.7) into (3.5) and by use of the definitions of a_n, b_n in (1.4), we obtain

$$\begin{aligned}\Lambda_2 V(M) \leq & \int_{M^n} e^{2u}\Big[n^2\big(|H|^2 + c - \frac{2}{n}\Delta u - \frac{n-2}{n}|\nabla u|^2\big) \\ & + (n-2)^2|\nabla u|^2\Big]dv_g + (na_n + b_n)\int_{M^n} Re^{2u}dv_g + \frac{n-4}{2}\int_{M^n} Qdv_g \\ = & n^2\int_{M^n} e^{2u}(|H|^2 + c)dv_g + (na_n + b_n)\int_{M^n} Re^{2u}dv_g \\ & + \frac{n-4}{2}\int_{M^n} Qdv_g + \int_{M^n} e^{2u}\Big((n-2)^2 - (n-2)n + 4n\Big)|\nabla u|^2 dv_g \\ = & n^2\int_{M^n} e^{2u}(|H|^2 + c)dv_g + 2(n+2)\int_{M^n} e^{2u}|\nabla u|^2 dv_g \\ & + \frac{n^2 - 2n - 4}{2(n-1)}\int_{M^n} Re^{2u}dv_g + \frac{n-4}{2}\int_{M^n} Qdv_g.\end{aligned} \tag{3.6}$$

From Gauss equation of $\phi : M^n \longrightarrow R^{n+p}(c)$

$$R = n(n-1)c + n|H|^2 - |\mathbf{h}|^2$$

and $|\mathbf{h}|^2 \geq n|H|^2$, we have

$$R \leq n(n-1)(|H|^2 + c). \tag{3.7}$$

The equality holds in (3.7) if and only if $\phi : M^n \longrightarrow R^{n+p}(c)$ is a total umbilical submanifold (see [6]).

By (3.6) and (3.7), we have

$$\begin{aligned}\Lambda_2 V(M) \leq & \frac{1}{2}n(n^2 - 4)\int_{M^n} e^{2u}(|H|^2 + c)dv_g + \frac{n-4}{2}\int_{M^n} Qdv_g \\ & + 2(n+2)\int_{M^n} e^{2u}|\nabla u|^2 dv_g.\end{aligned} \tag{3.8}$$

From (2.13), we have

$$
\begin{aligned}
\Lambda_2 V(M^n) \leq & \frac{1}{2}n(n^2-4)\int_{M^n}\left(|H|^2+c\right)^2 dv_g + \frac{n-4}{2}\int_{M^n} Q dv_g \\
& - \left(\frac{1}{2}(n-6)(n^2-4)-2(n+2)\right)\int_{M^n} e^{2u}|\nabla u|^2 dv_g \\
= & \frac{1}{2}n(n^2-4)\int_{M^n}\left(|H|^2+c\right)^2 dv_g + \frac{n-4}{2}\int_{M^n} Q dv_g \\
& - \frac{1}{2}(n+2)(n^2-8n+8)\int_{M^n} e^{2u}|\nabla u|^2.
\end{aligned}
\tag{3.9}
$$

Therefore, the inequality (1.6) follows immediately from inequality (3.9) if $n \geq 7$.

If the equality holds in (1.6), all the inequalities become equalities from (3.1) to (3.9). From (3.9), we can get $\nabla u = 0$, i.e. $u = constant$. In this case, (2.15) becomes equality, and then we can infer $\tilde{H} = 0$. (2.7) imply

$$|H|^2 + c = e^{2u} = constant. \tag{3.10}$$

Equality case in (3.7) give us $|\mathbf{h}|^2 = n|H|^2$, that is,

$$h^\alpha_{ij} = H^\alpha \delta_{ij}, \tag{3.11}$$

i.e., $\phi(M^n)$ is a totally umbilical submanifold in $R^{n+p}(c)$ (in [12], also called a geodesic sphere).

From (3.11)and Gauss equation of ϕ, we have

$$
\begin{aligned}
R_{ijkl} &= c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + h^\alpha_{ik}h^\alpha_{jl} - h^\alpha_{il}h^\alpha_{jk} \\
&= c(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + H^\alpha H^\alpha \delta_{ik}\delta_{jl} - H^\alpha H^\alpha \delta_{il}\delta_{jk} \\
&= (|H|^2 + c)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \\
R_{ij} &= (n-1)(|H|^2+c)\delta_{ij}, \\
R &= n(n-1)(|H|^2+c).
\end{aligned}
\tag{3.12}
$$

where R_{ijkl}, R_{ij} and R are the components of Riemannian curvature tensor, the Ricci tensor and scalar curvature of M^n, respectively.

By the definition of Q-curvature in (1.3), we have by use of (1.4)

$$
\begin{aligned}
Q &= -\frac{2}{(n-2)^2}|Ric|^2 + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}R^2 \\
&= -\frac{2}{(n-2)^2}(n-1)^2(|H|^2+c)^2\delta_{ik}\delta_{ik} \\
&\quad + \frac{n^3-4n^2+16n-16}{8(n-1)^2(n-2)^2}n^2(n-1)^2(|H|^2+c)^2 \\
&= \frac{1}{8}n(n^2-4)\left(|H|^2+c\right)^2.
\end{aligned}
\tag{3.13}
$$

Therefore, for the equality case in (1.6), we have

$$\Lambda_2 = \frac{1}{16} n(n+4)(n^2-4)(|H|^2+c)^2. \tag{3.14}$$

From (3.14), we have

$$|H|^2 + c = 4\sqrt{\frac{\Lambda_2}{n(n+4)(n^2-4)}}. \tag{3.15}$$

Therefore, from (3.12) and (3.15), we deduce that $\phi(M)$ is a geodesic sphere $\mathbb{S}^n(r_c)$ with radius r_c defined by (1.7).

Conversely, suppose that $\phi(M)$ is a geodesic sphere $\mathbb{S}^m(r_c)$ with radius r_c defined by (1.7) in space form $R^{n+p}(c)$. It is easily deduced that the section curvature

$$R_{ijij} = 4\sqrt{\frac{\Lambda_2}{n(n+4)(n^2-4)}}, \quad i \neq j. \tag{3.16}$$

From (3.12), we obtain (3.15). Therefore the equality holds in (1.6). We complete the proof of Theorem 1.1.

Remark 3.1. If we assume that the scalar curvature R is nonnegative, from (3.7) we have

$$R^2 \leq n^2(n-1)^2(|H|^2+c)^2. \tag{3.17}$$

Inserting (3.17) into (1.8), we have under $R \geq 0$ and the same assumptions as in the theorem 1.1

$$\Lambda_2 V(M) \leq \frac{1}{16} n(n+4)(n^2-4) \int_{M^n} (|H|^2+c)^2. \tag{3.18}$$

Moreover, the equality holds if and only if M^n is an n-dimensional geodesic sphere.

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